厦門大學信息学院
School of Informatics Xiamen University
(特色化示范性软件学院)
(National Characteristic Demonstration Software School)

《离散数学》

Chapter 10：Graphs（I）

王晓黎

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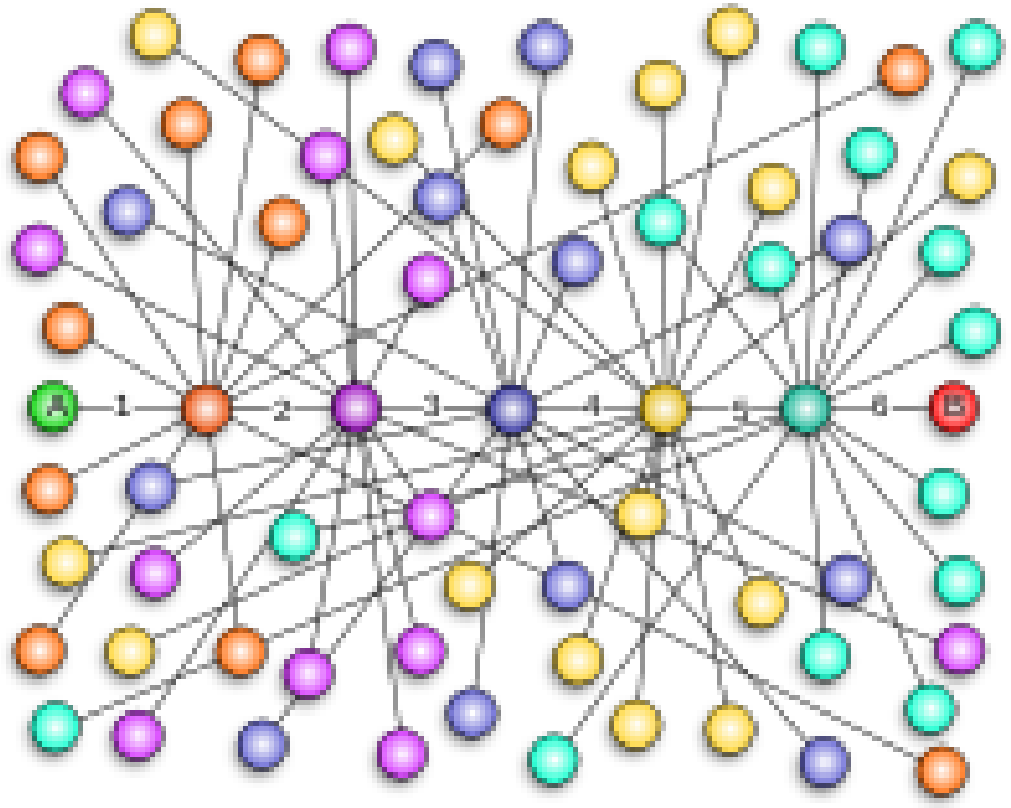
Outline

- Small World Experiment
- Graphs and Graph Models
- Graph Terminology and Special Types of Graphs
- Representing Graphs and Graph Isomorphism (图同构)
- Connectivity (连接性)
- Euler and Hamiltonian Graphs (欧拉图、哈密顿图)

Small-World Phenomenon (小世界理论)

- When we look at large social network with thousands of nodes, we find that distances are generally quite short, often less than 10
  - This is called the Small-World phenomenon



Milgram's Small World Experiment

- 296 letters are mailed out for a person in U.S.
- 64 reached, and the average path length fell around five and a half or six

Problems in the Experiment

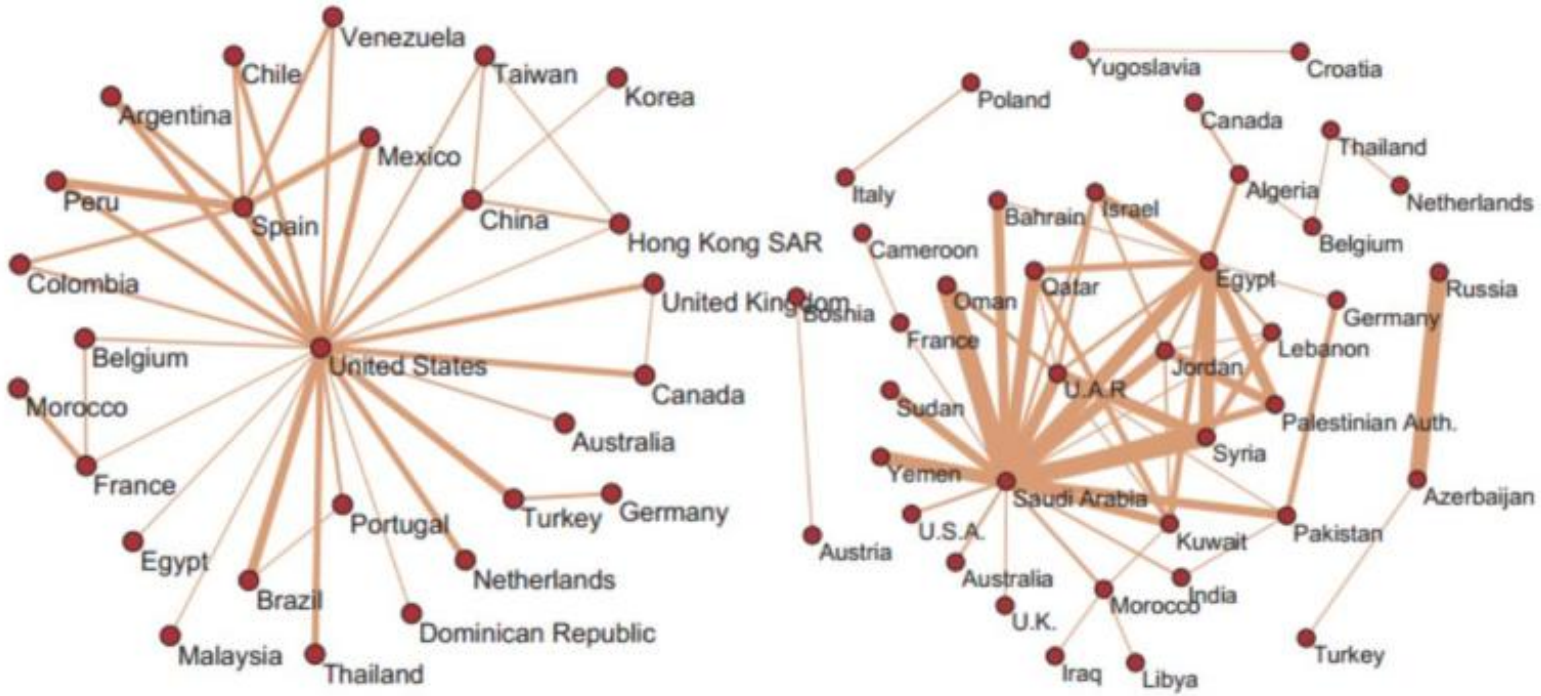
- Suffers from selection and non-response bias
  - Only a small number of letters reached its destination
    - 64 of 296
    - 24 of 160
- Why?
  - People often refuse to pass the letter forward

How to improve the experiments?

Use a social graph (社交图) to model social relationships between humans

A New Experiment

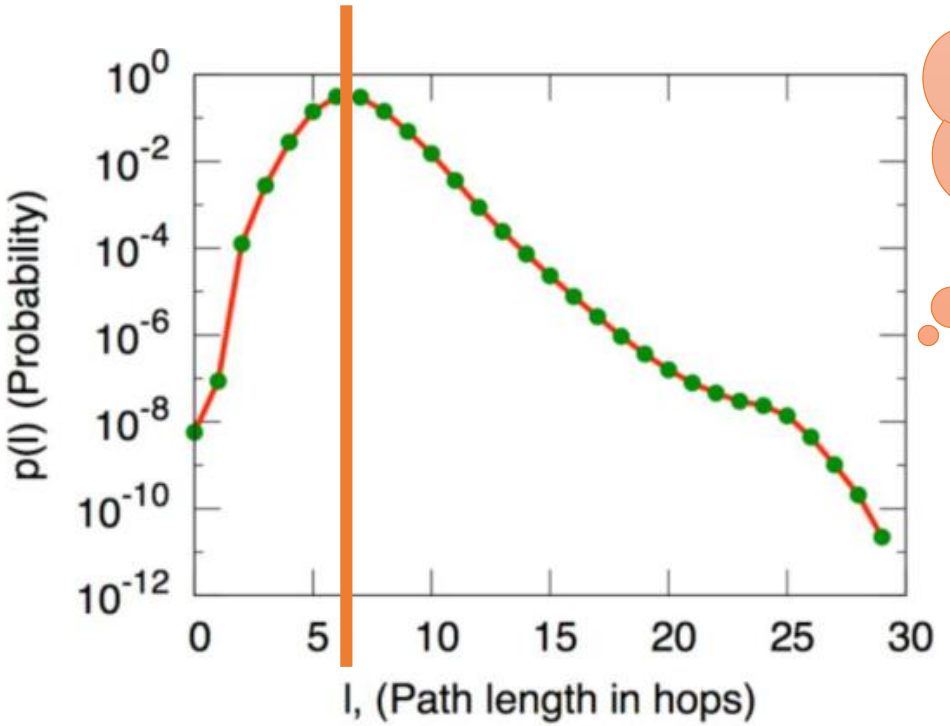
- A large-scale study was done by Leskovec & Horvitz (2008) in Microsoft Instant Messenger network
  - Analyzed 240 million active user accounts for a month



https://arxiv.org/abs/0803.0939

The Results

- The small world phenomenon has been replicated in this study
  - Estimated Average Distance = 6.6



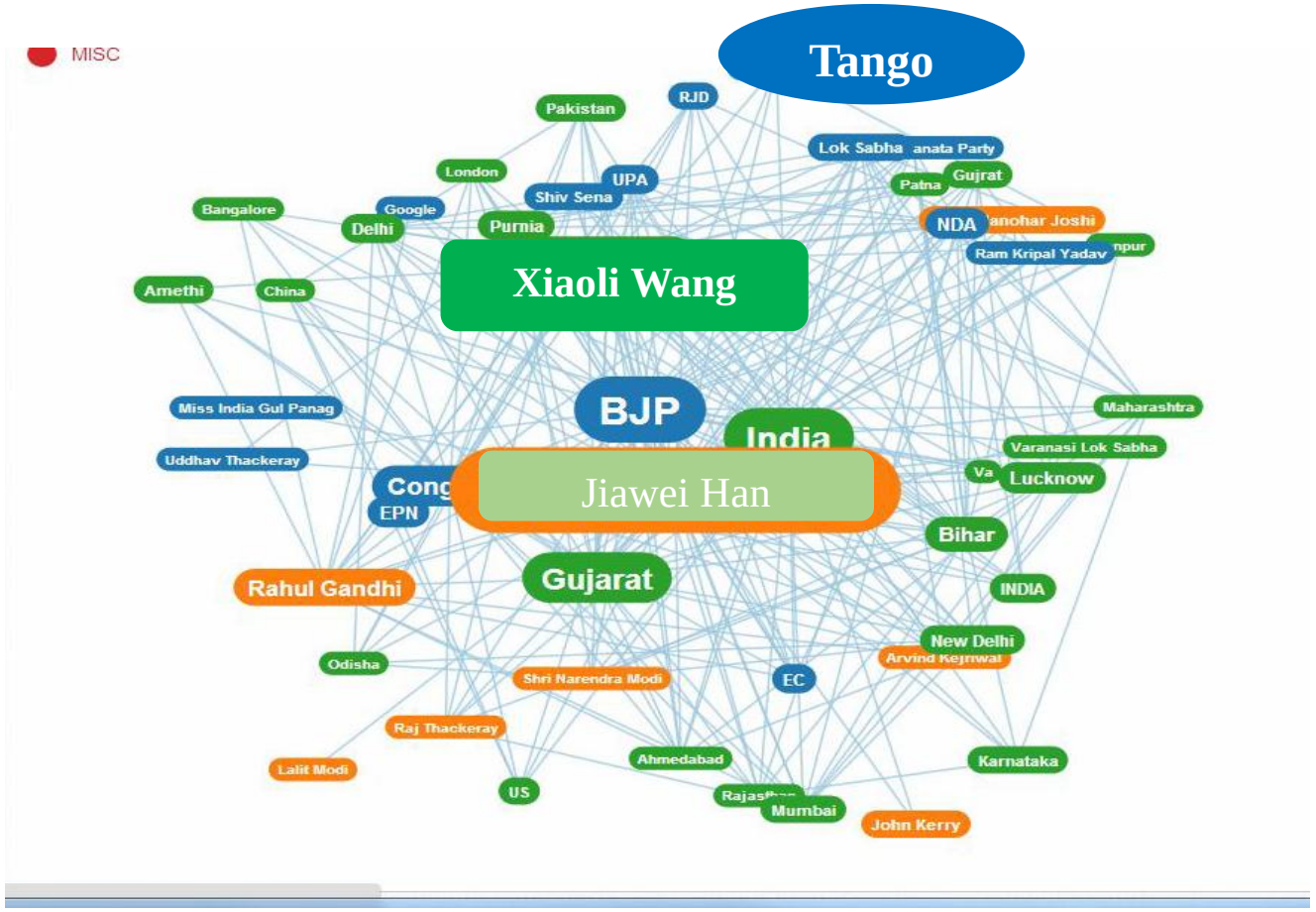
Why?

Small-World Phenomenon (Continued)

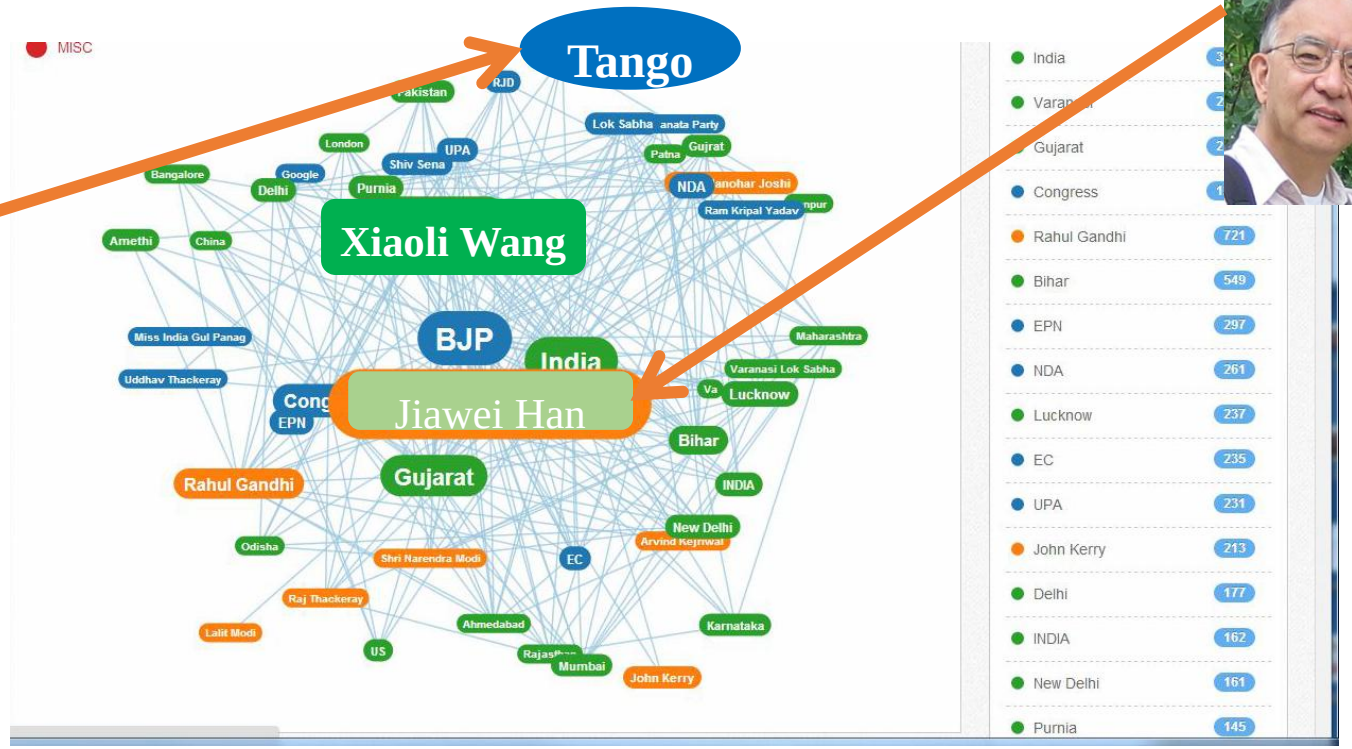
- In fact, a distance of 6 can socially scaled to a huge number
  - Suppose everyone has on average 100 acquaintances and there is little overlap between acquaintanceships
    - Me: 1
    - Acquaintances: 100
    - Acquaintances at distance 2: 100^2=10,000
    - Acquaintances at distance 3: 100^3=1,000,000
    - Acquaintances at distance 4: 100^4=100,000,000
    - Acquaintances at distance 5: 100^5=10,000,000,000

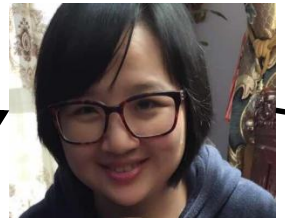
The application of social graph

- Build a social graph from social network systems



The application of social graph (Continued)





Two questions to be answered：

What is graph?

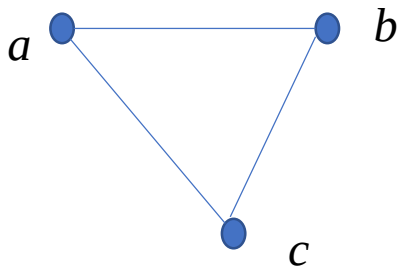
What is social graph?

Graphs and Graph Models

- Introduction to Graphs
- Graph Taxonomy
- Graph Models

Graphs

- **Definition**
  - A *graph* $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *edges*
    - Each edge has either one or two vertices associated with it, called its *endpoints* (端点)
    - An edge is said to *connect* its endpoints



Example:

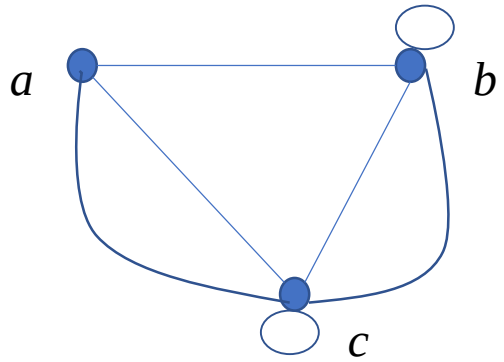
This is a graph with three vertices and three edges

- **Remarks**
  - A graph with an infinite vertex set is called an infinite graph
  - A graph with a finite vertex set is called a finite graph
  - We restrict our attention to finite graphs

Some Terminology

- In a *simple graph* (简单图) each edge connects two different vertices and no two edges connect the same pair of vertices
- *Multigraphs* (多重图) may have multiple edges connecting the same two vertices. When m different edges connect the vertices u and v, we say that $\{u,v\}$ is an edge of *multiplicity* (重数) m
- An edge that connects a vertex to itself is called a *loop* (环)
- A *pseudograph* (伪图) may include loops, as well as multiple edges connecting the same pair of vertices

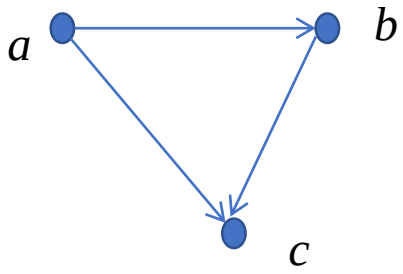
Example:
This pseudograph has both multiple edges and a loop



Remark: There is no standard terminology for graph theory. So, it is crucial that you understand the terminology being used whenever you read material about graphs.

Directed Graphs

- **Definition**
  - An *directed graph* (or *digraph*) (有向图) $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *directed edges* (or *arcs*) (有向边)
    - Each edge is associated with an ordered pair of vertices
    - The directed edge associated with the ordered pair (u,v) is said to *start at* u and *end at* v



Example:

This is a directed graph with three vertices and three edges

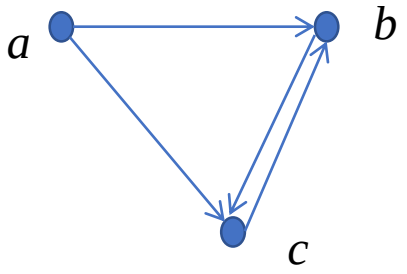
- **Remarks**
  - Graphs where the endpoints of an edge are not ordered are said to be *undirected graphs* (无向图)

Some Terminology (continued)

- A *simple directed graph* has no loops and no multiple edges

Example:

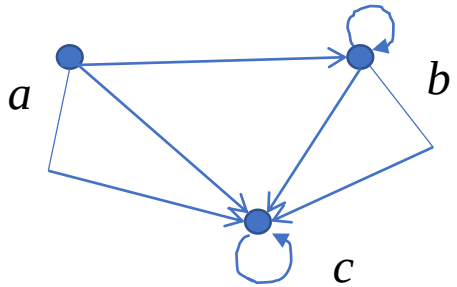
This is a directed graph with three vertices and four edges



- A *directed multigraph* may have multiple directed edges. When there are m directed edges from the vertex u to the vertex v, we say that (u,v) is an edge of *multiplicity* m

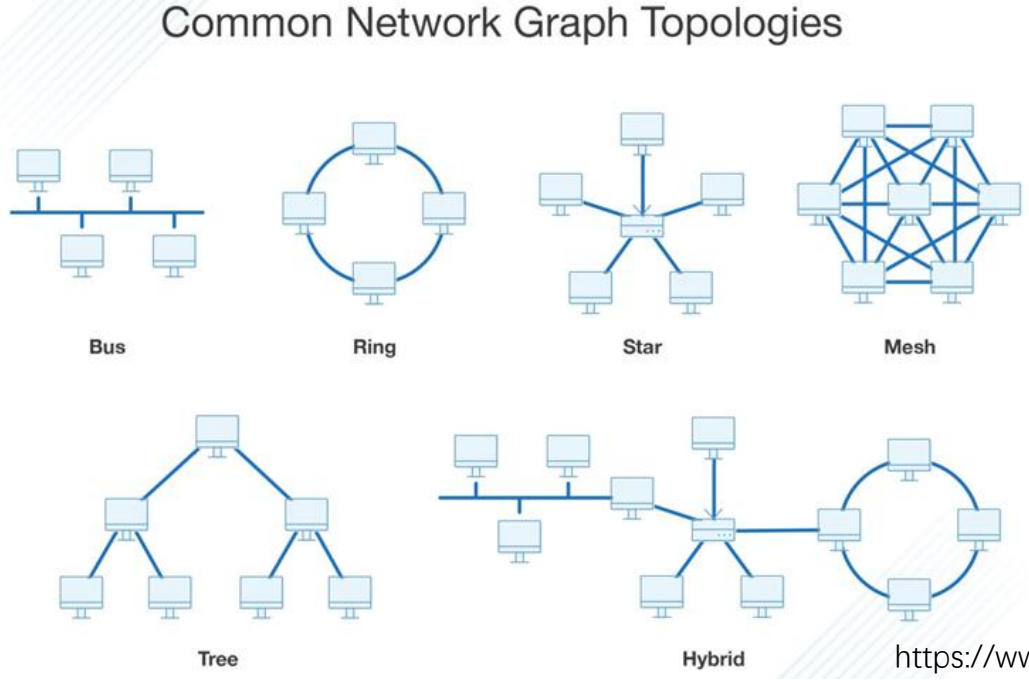
Example:

In this directed multigraph the multiplicity of (a,b) is 1 and the multiplicity of (b,c) is 2



Graph Models: Computer Networks

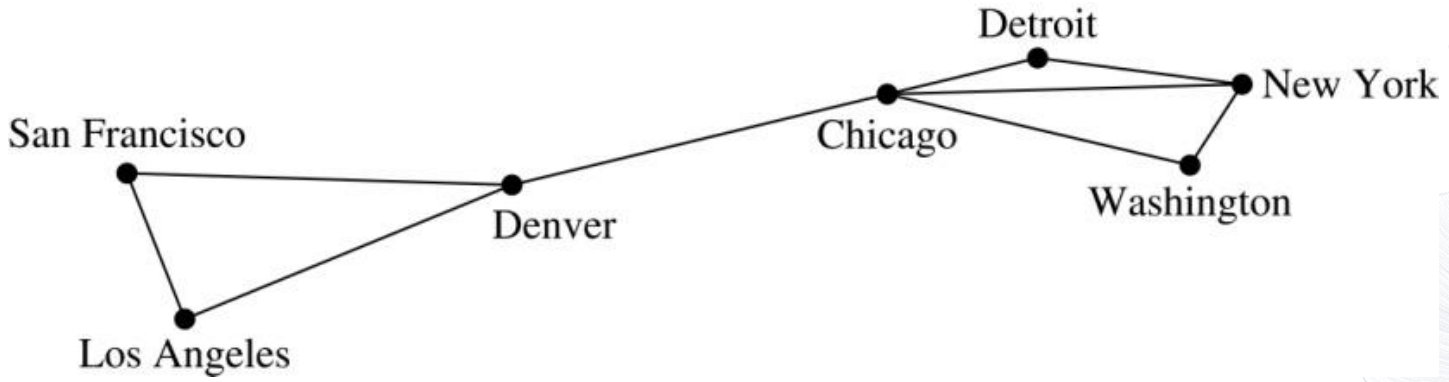
- When we build a graph model, we use the appropriate type of graph to capture the important features of the application
- We illustrate this process using graph models of different types of computer networks
  - In all these graph models, the vertices represent data centers and the edges represent communication links



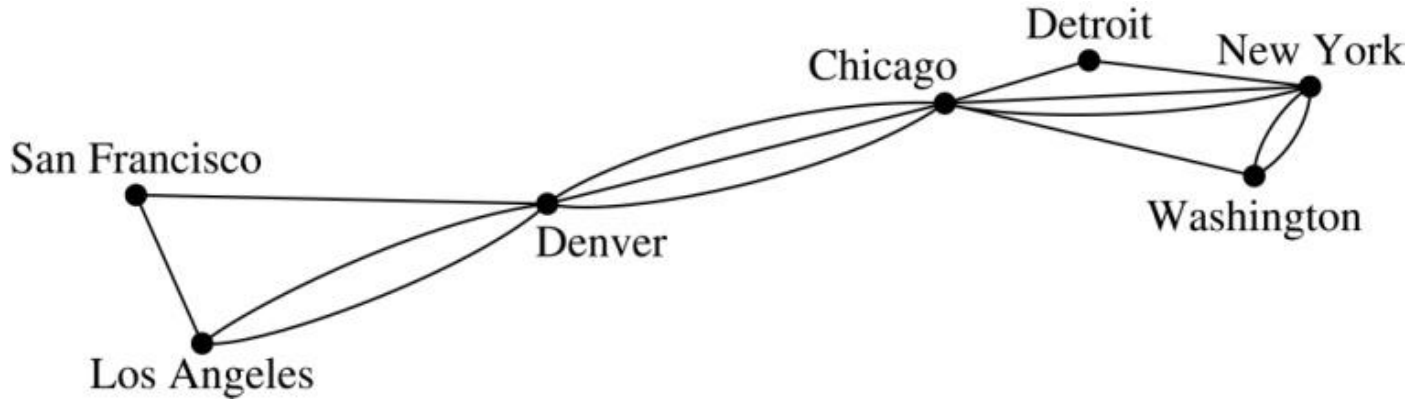
https://www.dnsstuff.com/network-graphs

Graph Models: Computer Networks

- To model a computer network where we are only concerned whether two data centers are connected by a communications link, we use a simple graph

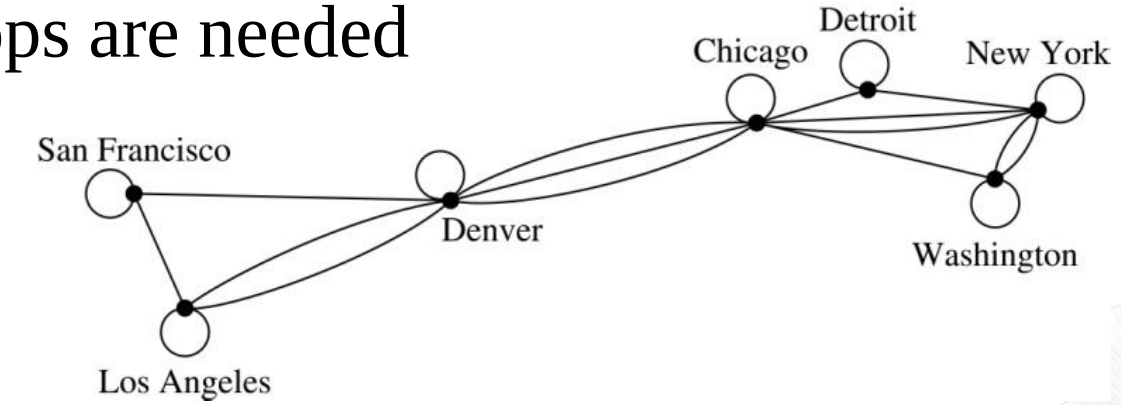


- To model a computer network where we care about the number of links between data centers, we use a multigraph

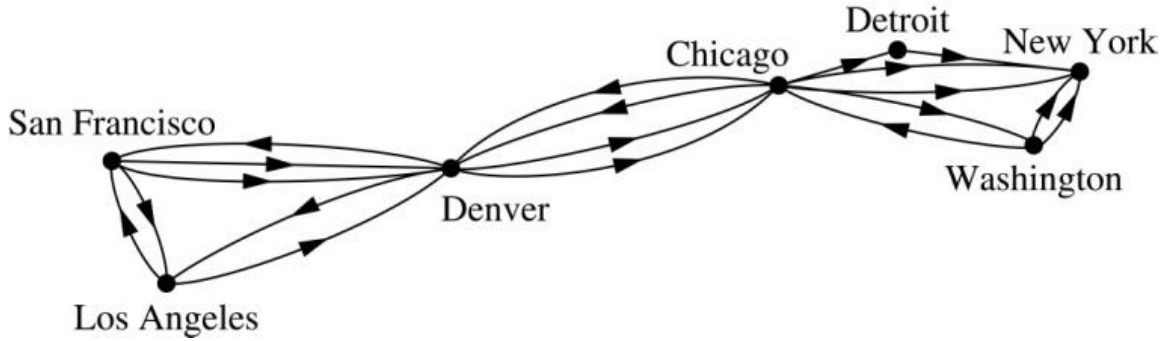


Graph Models: Computer Networks (continued)

- To model a computer network with diagnostic links at data centers, we use a pseudograph, as loops are needed



- To model a network with multiple one-way links, we use a directed multigraph
  - Note that we could use a directed graph without multiple edges if we only care whether there is at least one link from a data center to another data center



Graph Terminology: Summary

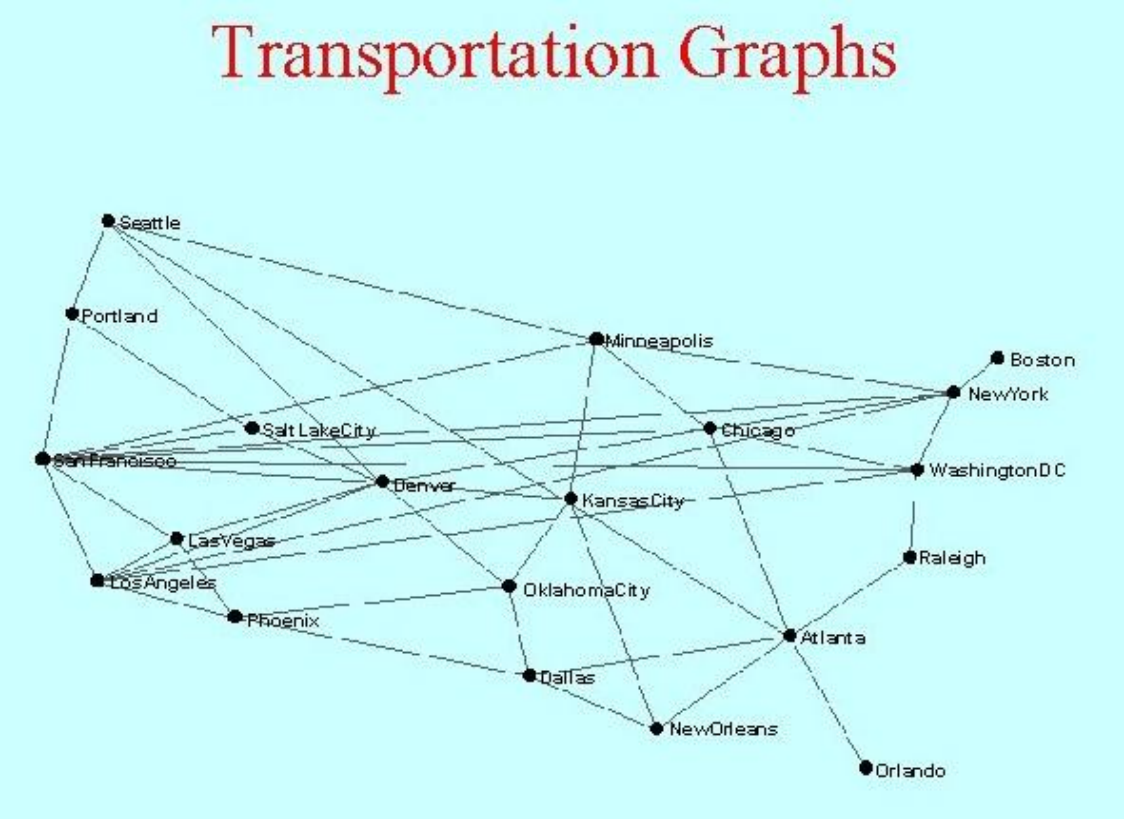
- To understand the structure of a graph and to build a graph model, we ask these questions:
  - Are the edges of the graph undirected or directed (or both)?
  - If the edges are undirected, are multiple edges present that connect the same pair of vertices? If the edges are directed, are multiple directed edges present?
  - Are loops present?

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Other Applications of Graphs

- We will illustrate how graph theory can be used in models of:
  - Transportation networks



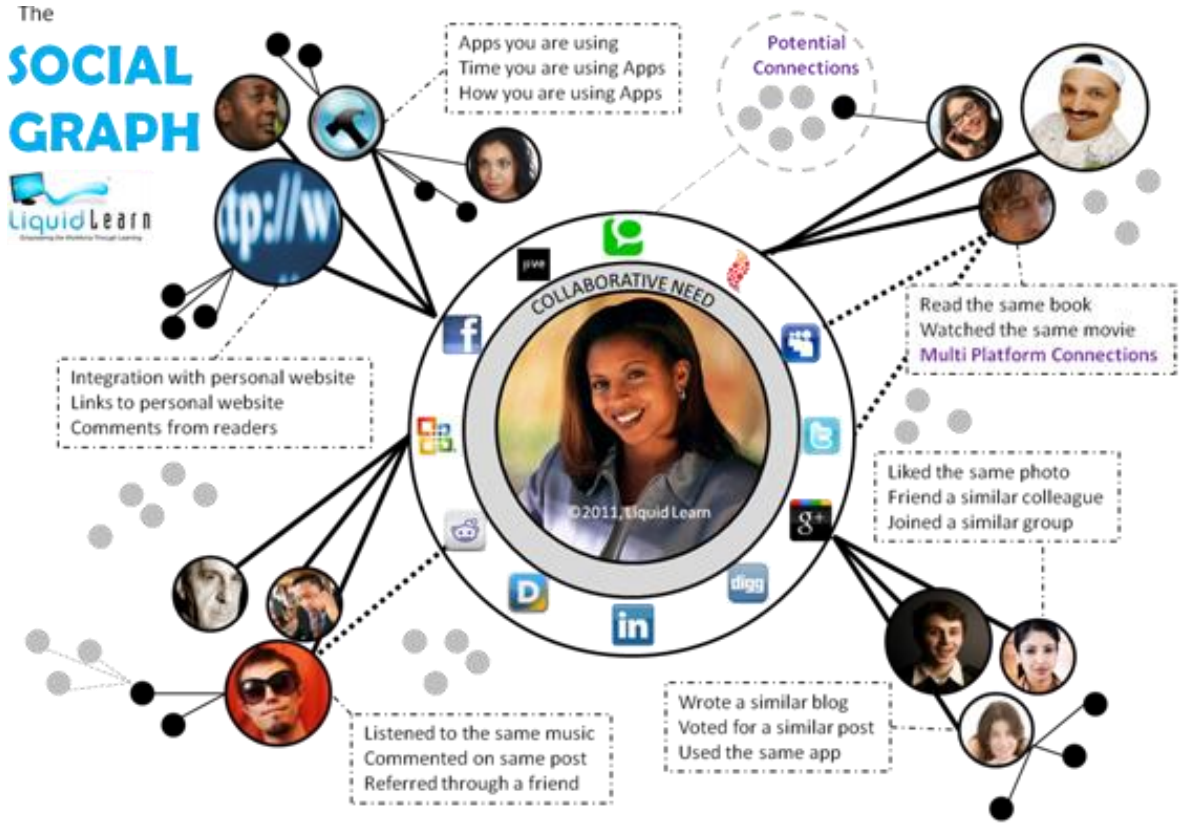
https://slidetodoc.com/graphs-eric-roberts-cs-106-b-february-25/

Other Applications of Graphs

- We will illustrate how graph theory can be used in models of:
  - Social networks
  - Communications networks
  - Information networks
  - Software design
  - Transportation networks
  - Biological networks
  - …
- It's a challenge to find a subject to which graph theory has not yet been applied. Can you find an area without applications of graph theory?

Graph Models: Social Networks

- Graphs can be used to model social structures based on different kinds of relationships between people or groups
- In a *social network*, vertices represent individuals or organizations and edges represent relationships between them

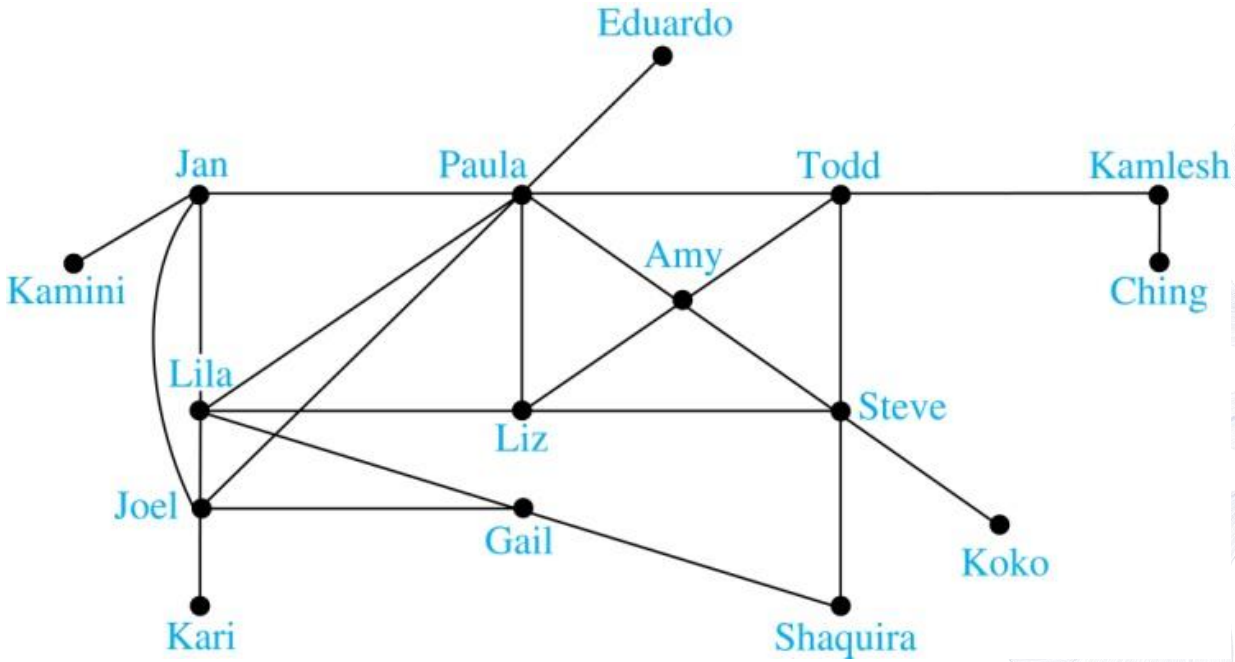


Graph Models: Social Networks

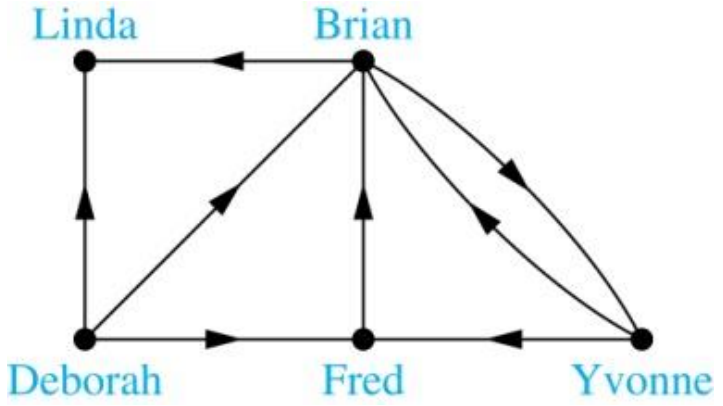
- Graphs can be used to model social structures based on different kinds of relationships between people or groups
- In a *social network*, vertices represent individuals or organizations and edges represent relationships between them
- Useful graph models of social networks include:
  - *friendship graphs* - undirected graphs where two people are connected if they are friends (in the real world, on Facebook, or in a particular virtual world, and so on.)
  - *collaboration graphs* - undirected graphs where two people are connected if they collaborate in a specific way
  - *influence graphs* - directed graphs where there is an edge from one person to another if the first person can influence the second person

Graph Models: Social Networks (continued)

Example: A friendship graph where two people are connected if they are Facebook friends

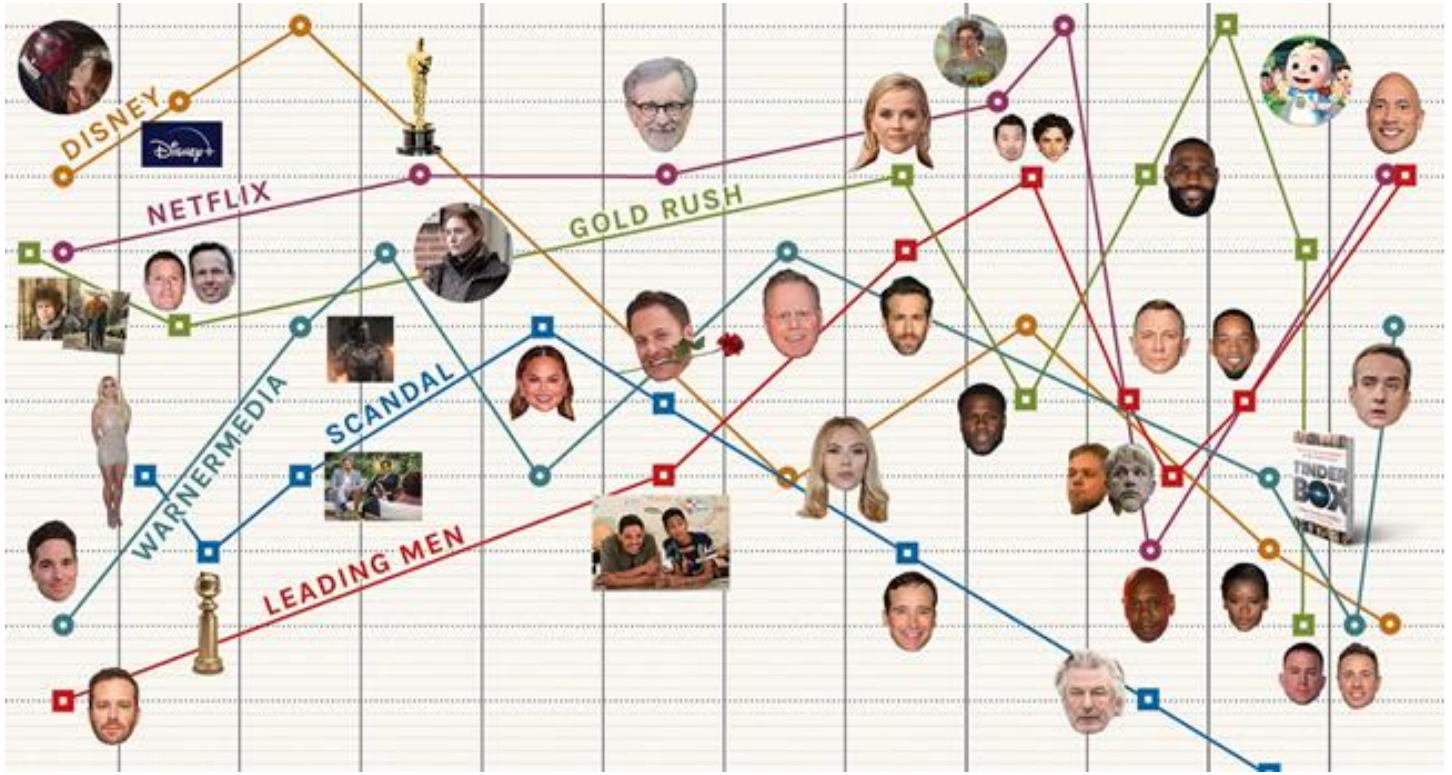


Example: An influence graph



Examples of Collaboration Graphs

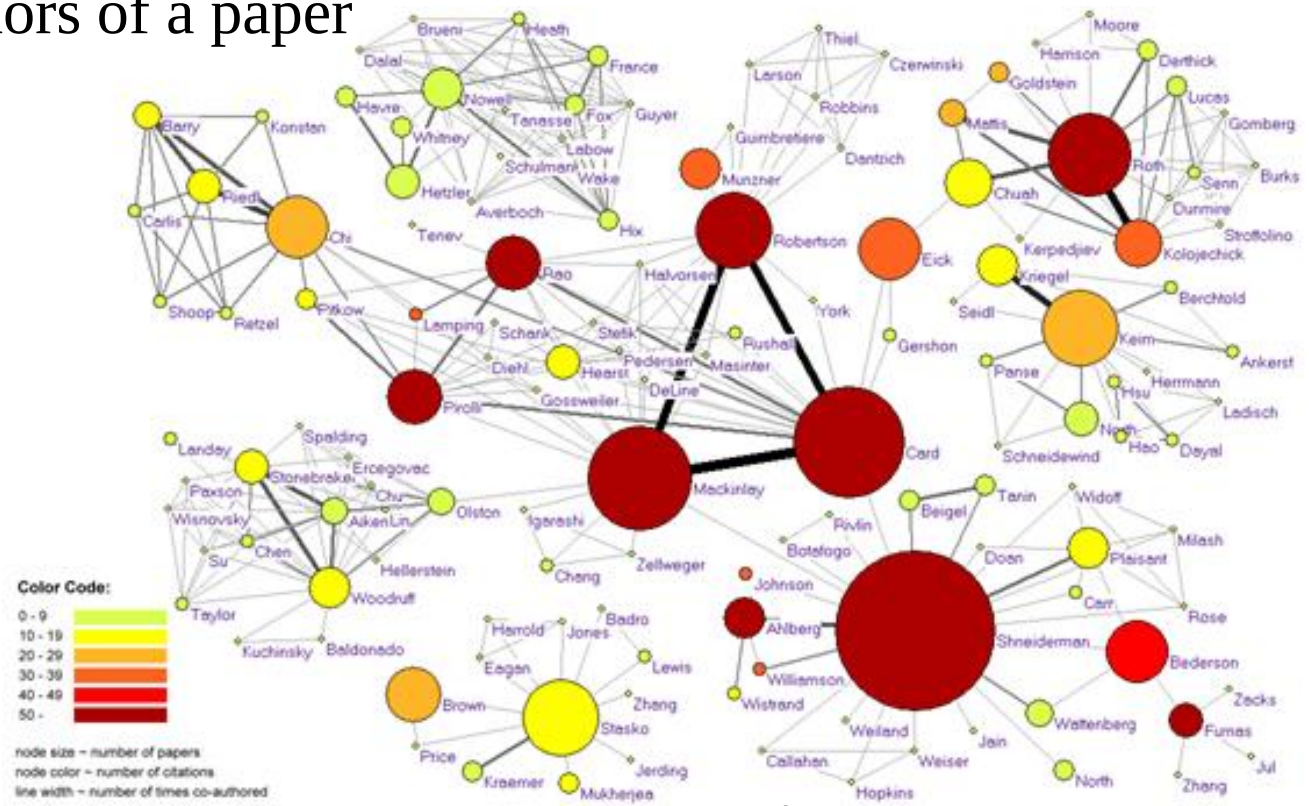
- The *Hollywood graph* models the collaboration of actors in films
  - We represent actors by vertices and we connect two vertices if the actors they represent have appeared in the same movie



https://www.hollywoodreporter.com/business/business-news/hollywood-2021-highs-and-lower-lows-1235062740/

Examples of Collaboration Graphs

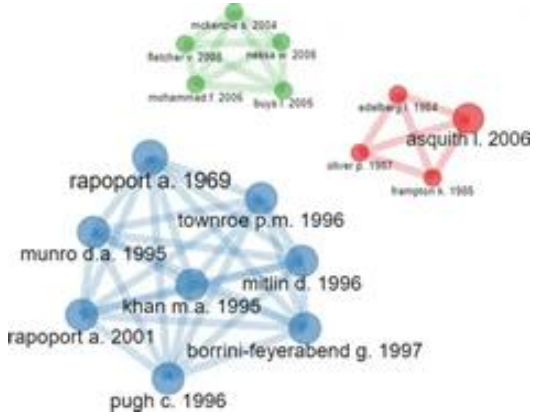
- An *academic collaboration graph* models the collaboration of researchers who have jointly written a paper in a particular subject
  - We represent researchers in a particular academic discipline using vertices
  - We connect the vertices representing two researchers in this discipline if they are coauthors of a paper



https://cns.iu.edu/2004-InfoVis.html

Applications to Information Networks

- Graphs can be used to model different types of networks that link different types of information
- In a *web graph*, web pages are represented by vertices and links are represented by directed edges
  - A web graph models the web at a particular time
- In a *citation network*:
  - Research papers in a particular discipline are represented by vertices
  - When a paper cites a second paper as a reference, there is an edge from the vertex representing this paper to the vertex representing the second paper



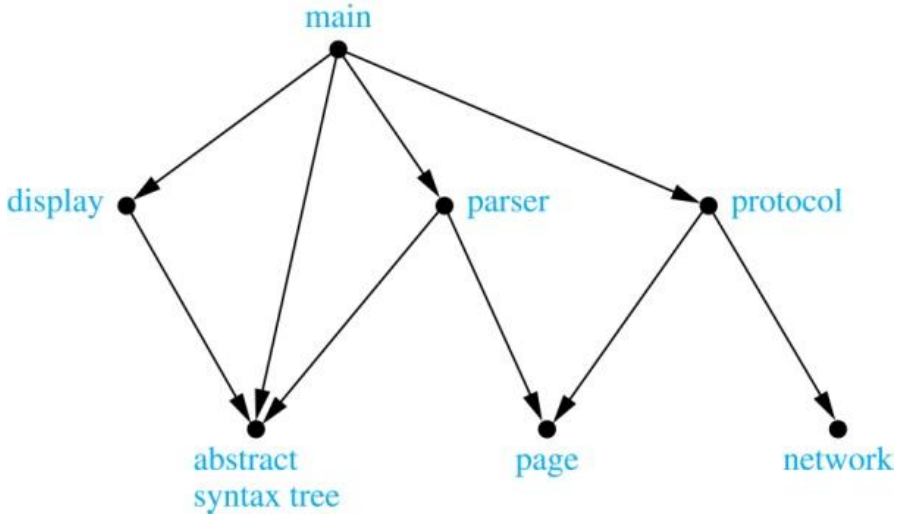
Transportation Graphs

- Graph models are extensively used in the study of transportation networks
- Airline networks can be modeled using directed multigraphs where
  - airports are represented by vertices
  - each flight is represented by a directed edge from the vertex representing the departure airport to the vertex representing the destination airport
- Road networks can be modeled using graphs where
  - vertices represent intersections and edges represent roads
  - undirected edges represent two-way roads and directed edges represent one-way roads

Software Design Applications

- Graph models are extensively used in software design
- When a top-down approach is used to design software, the system is divided into modules, each performing a specific task
- We use a *module dependency graph* to represent the dependency between these modules. These dependencies need to be understood before coding can be done
  - In a module dependency graph vertices represent software modules and there is an edge from one module to another if the second module depends on the first

Example: The dependencies between the seven modules in the design of a web browser are represented by this module dependency graph



Software Design Applications (*continued*)

- We can use a directed graph called a *precedence graph* to represent which statements must have already been executed before we execute each statement
  - Vertices represent statements in a computer program
  - There is a directed edge from a vertex to a second vertex if the second vertex cannot be executed before the first

Example: This precedence graph shows which statements must already have been executed before we can execute each of the six statements in the program

S_1 $a := 0$

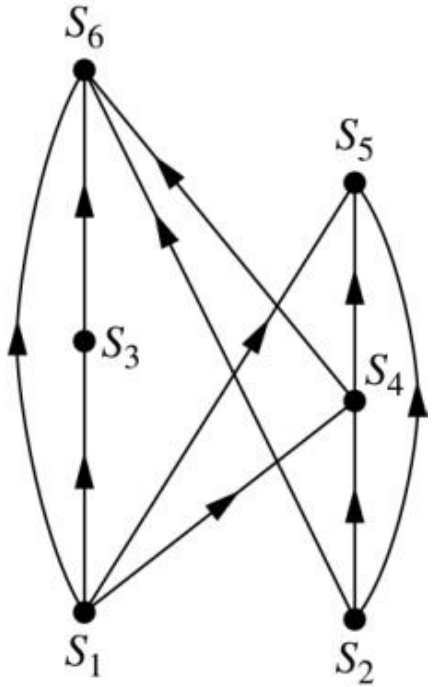
S_2 $b := 1$

S_3 $c := a + 1$

S_4 $d := b + a$

S_5 $e := d + 1$

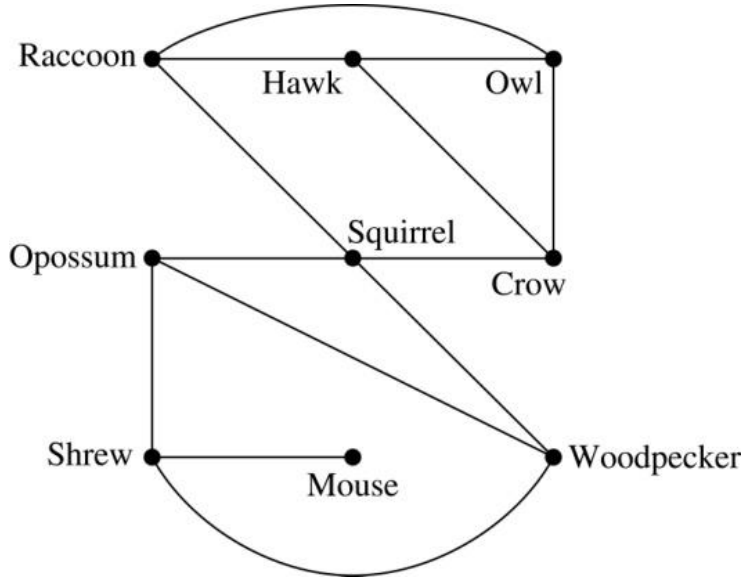
S_6 $e := c + d$



Biological Applications

- Graph models are used extensively in many areas of the biological science
- *Niche overlap graphs* model competition between species in an ecosystem
  - Vertices represent species and an edge connects two vertices when they represent species who compete for food resources

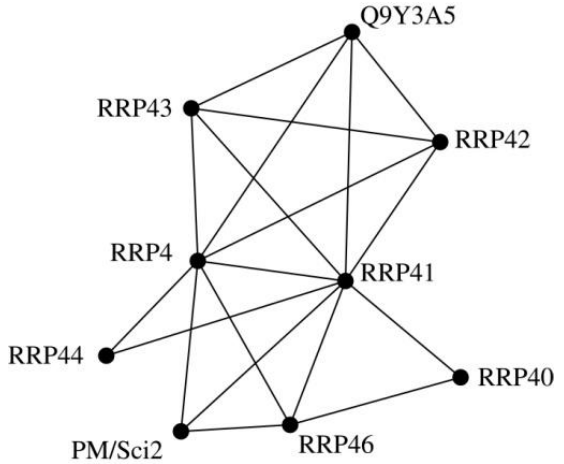
Example: This is the niche overlap graph for a forest ecosystem with nine species



Biological Applications (*continued*)

- We can model the interaction of proteins in a cell using a *protein interaction network*
- In a *protein interaction graph*, vertices represent proteins and vertices are connected by an edge if the proteins they represent interact
- Protein interaction graphs can be huge and can contain more than 100,000 vertices, each representing a different protein, and more than 1,000,000 edges, each representing an interaction between proteins
- Protein interaction graphs are often split into smaller graphs, called *modules*, which represent the interactions between proteins involved in a particular function

Example: This is a module of the protein interaction graph of proteins that degrade RNA in a human cell



Graph Terminology and Special Types of Graphs

- Basic Terminology
- Some Special Types of Graphs
- Bipartite Graphs
- Bipartite Graphs and Matchings
- Some Applications of Special Types of Graphs
- New Graphs from Old

Basic Terminology

Definition 1. Two vertices *u*, *v* in an undirected graph *G* are called *adjacent* (or *neighbors*) in *G* if there is an edge *e* between *u* and *v*. Such an edge *e* is called *incident with* the vertices *u* and *v* and *e* is said to *connect* *u* and *v*

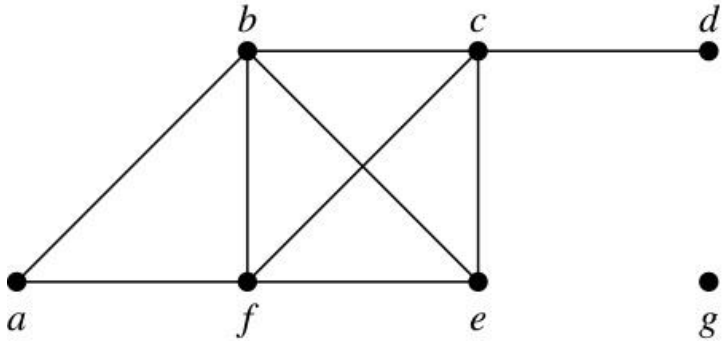
Definition 2. The set of all neighbors of a vertex *v* of $G = (V, E)$, denoted by $N(v)$, is called the *neighborhood* of *v*. If *A* is a subset of *V*, we denote by $N(A)$ the set of all vertices in *G* that are adjacent to at least one vertex in *A*
So,

$$N(A) = \bigcup_{v \in A} N(v).$$

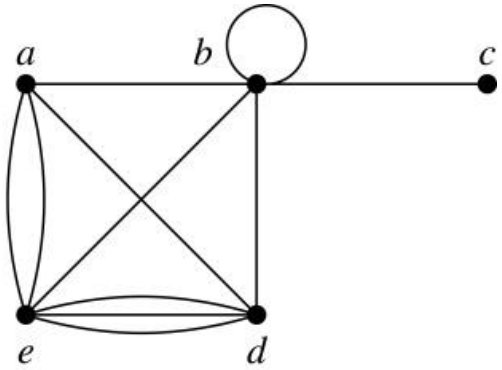
Definition 3. The *degree of a vertex in a undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex *v* is denoted by $\deg(v)$

Degrees and Neighborhoods of Vertices

Example: What are the degrees and neighborhoods of the vertices in the graphs G and H?



G



H

Solution:

G: $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$,
$\deg(e) = 3$, $\deg(g) = 0$.
$N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c\}$,
$N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, $N(g) = \varnothing$.

H: $\deg(a) = 4$, $\deg(b) = \deg(e) = 6$, $\deg(c) = 1$, $\deg(d) = 5$.
$N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$,
$N(d) = \{a, b, e\}$, $N(e) = \{a, b, d\}$.

Degrees of Vertices

Theorem 1 (*Handshaking Theorem*): If $G = (V,E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} \deg(v)$$

Proof:

Each edge contributes twice to the degree count of all vertices. Hence, both the left-hand and right-hand sides of this equation equal twice the number of edges. ◀

Think about the graph where vertices represent the people at a party and an edge connects two people who have shaken hands

Handshaking Theorem

We now give two examples illustrating the usefulness of the handshaking theorem

Example: How many edges are there in a graph with 10 vertices of degree six?

Solution: Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, the handshaking theorem tells us that $2m = 60$. So the number of edges $m = 30$.

Example: If a graph has 5 vertices, can each vertex have degree 3?

Solution: This is not possible by the handshaking thorem, because the sum of the degrees of the vertices $3 \cdot 5 = 15$ is odd.

Degree of Vertices (*continued*)

Theorem 2: An undirected graph has an even number of vertices of odd degree

***Proof*:** Let V_1 be the vertices of even degree and V_2 be the vertices of odd degree in an undirected graph $G = (V, E)$ with m edges. Then

even $\longrightarrow$ $$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v).$$

must be even since $\deg(v)$ is even for each $v \in V_1$

This sum must be even because $2m$ is even and the sum of the degrees of the vertices of even degrees is also even. Because this is the sum of the degrees of all vertices of odd degree in the graph, there must be an even number of such vertices.

Directed Graphs

Recall the definition of a directed graph.

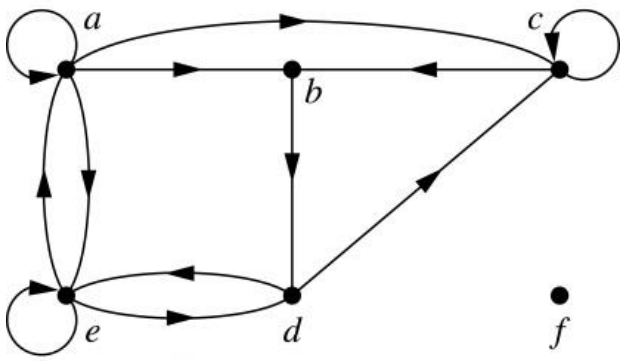
Definition: An *directed graph* $G = (V, E)$ consists of V, a nonempty set of *vertices* (or *nodes*), and E, a set of *directed edges* or *arcs*. Each edge is an ordered pair of vertices. The directed edge (u,v) is said to start at u and end at v

Definition: Let (u,v) be an edge in G. Then u is the *initial vertex* of this edge and is *adjacent to* v and v is the *terminal* (or *end*) *vertex* of this edge and is *adjacent from* u. The initial and terminal vertices of a loop are the same

Directed Graphs (*continued*)

Definition: The *in-degree of a vertex* v, denoted $deg^-(v)$, is the number of edges which terminate at v. The *out-degree of* v, denoted $deg^+(v)$, is the number of edges with v as their initial vertex. Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex

Example: In the graph G we have



G

$\deg^-(a) = 2$, $\deg^-(b) = 2$, $\deg^-(c) = 3$, $\deg^-(d) = 2$, $\deg^-(e) = 3$, $\deg^-(f) = 0$.

$\deg^+(a) = 4$, $\deg^+(b) = 1$, $\deg^+(c) = 2$, $\deg^+(d) = 2$, $\deg^+(e) = 3$, $\deg^+(f) = 0$.

Directed Graphs (*continued*)

Theorem 3: Let $G = (V, E)$ be a graph with directed edges. Then:

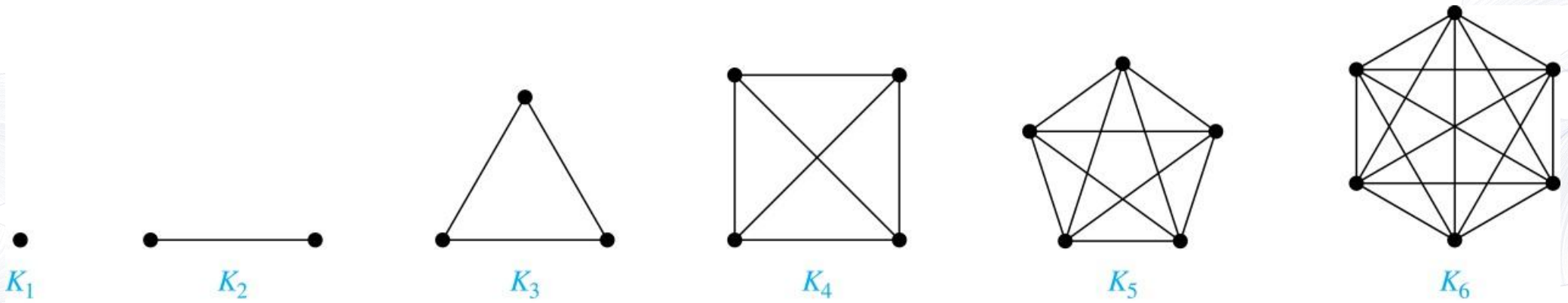
$$|E| = \sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v).$$

Proof: The first sum counts the number of outgoing edges over all vertices and the second sum counts the number of incoming edges over all vertices. It follows that both sums equal the number of edges in the graph.

◀

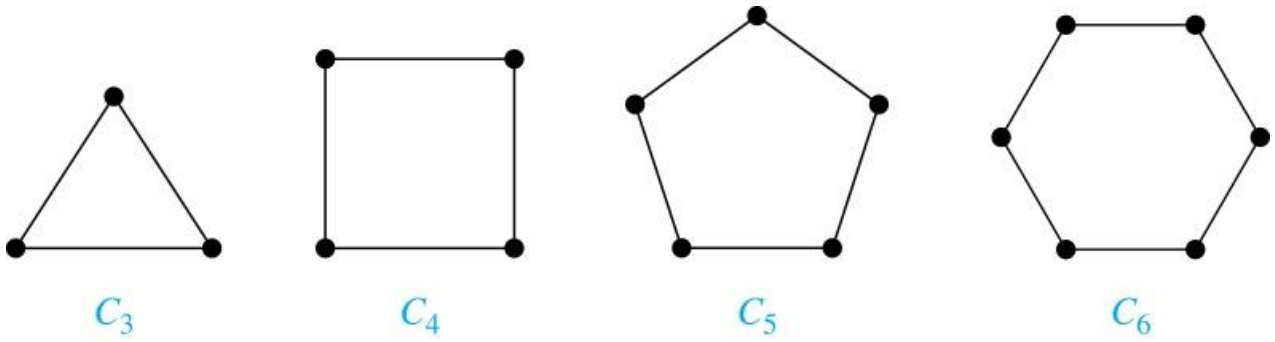
Special Types of Simple Graphs: Complete Graphs

A *complete graph on n vertices*, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices

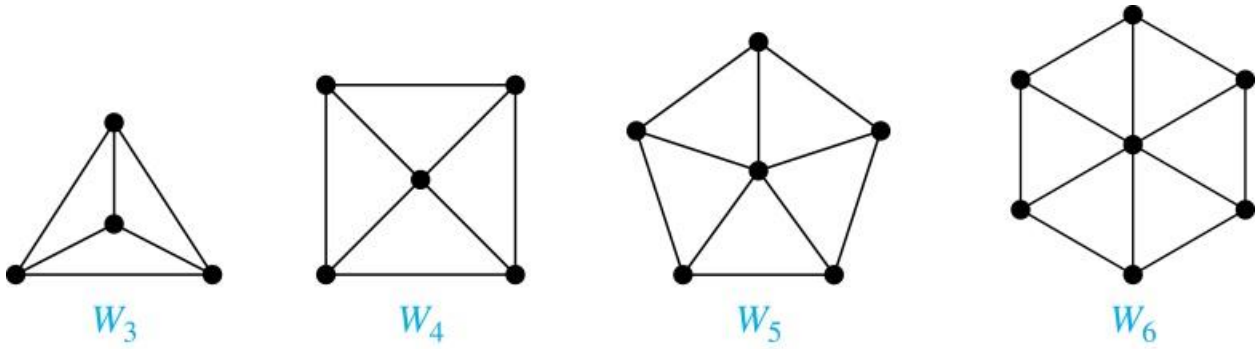


Special Types of Simple Graphs: Cycles and Wheels

A *cycle* C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \cdots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$

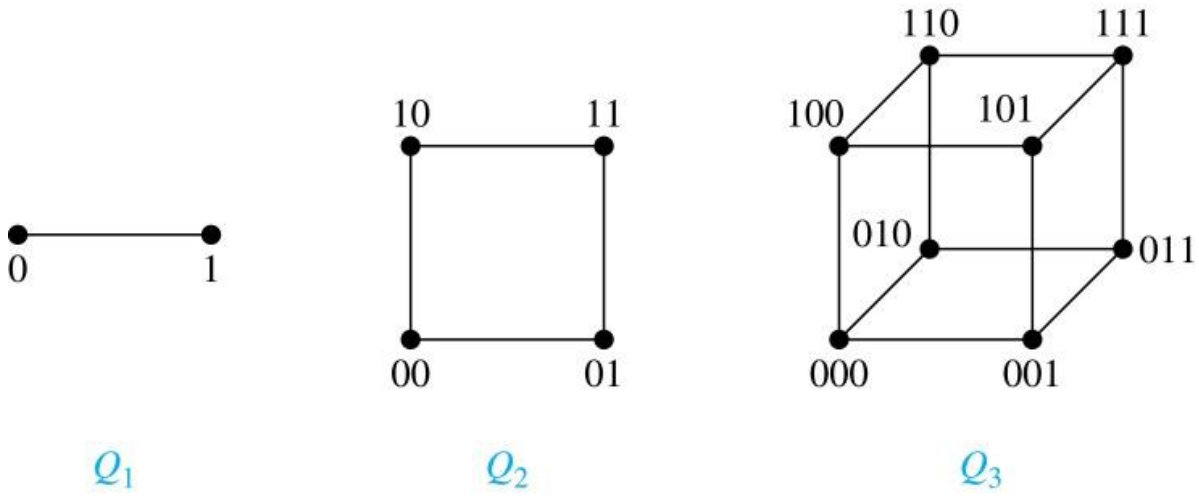


A *wheel* W_n is obtained by adding an additional vertex to a cycle C_n for $n \geq 3$ and connecting this new vertex to each of the n vertices in C_n by new edges



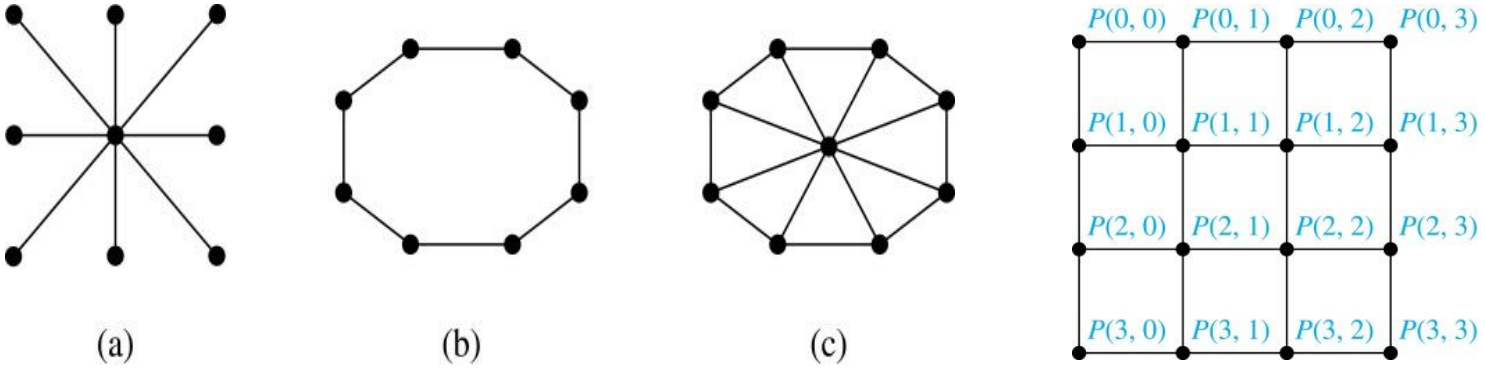
Special Types of Simple Graphs: *n*-Cubes

An *n-dimensional hypercube*, or *n-cube*, $\boldsymbol{Q_n}$, is a graph with 2^n vertices representing all bit strings of length *n*, where there is an edge between two vertices that differ in exactly one bit position



Special Types of Graphs and Computer Network Architecture

Various special graphs play an important role in the design of computer networks



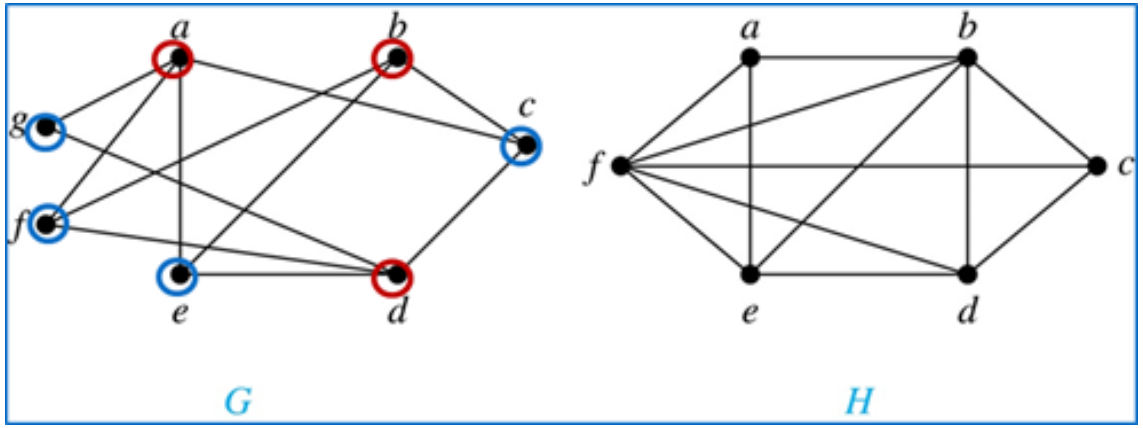
- Some local area networks use a *star topology*, which is a complete bipartite graph $K_{1,n}$,as shown in (a). All devices are connected to a central control device
- Other local networks are based on a *ring topology*, where each device is connected to exactly two others using C_n ,as illustrated in (b). Messages may be sent around the ring
- Others, as illustrated in (c), use a W_n-based topology, combining the features of a star topology and a ring topology
- Various special graphs also play a role in parallel processing where processors need to be interconnected as one processor may need the output generated by another
  - The *n-dimensional hypercube*, or *n-cube*, $\boldsymbol{Q_n}$, is a common way to connect processors in parallel, e.g., Intel Hypercube
  - Another common method is the *mesh* network, illustrated here for 16 processors

Bipartite Graphs

Definition: A simple graph G is bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2. In other words, there are no edges which connect two vertices in V_1 or in V_2

It is not hard to show that an equivalent definition of a bipartite graph is a graph where it is possible to color the vertices red or blue so that no two adjacent vertices are the same color

G is bipartite

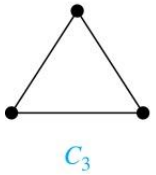


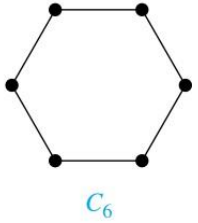
H is not bipartite since if we color a red, then the adjacent vertices f and b must both be blue.

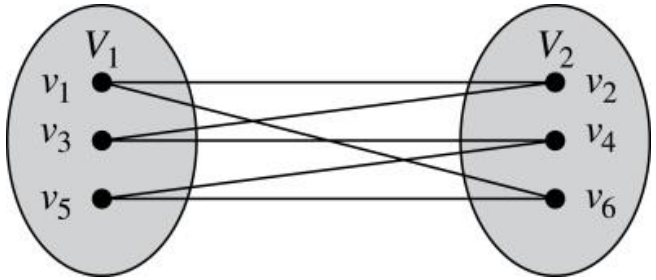
Bipartite Graphs (*continued*)

Example: Show that C_6 is bipartite.

Solution: We can partition the vertex set into $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ so that every edge of C_6 connects a vertex in V_1 and V_2 .







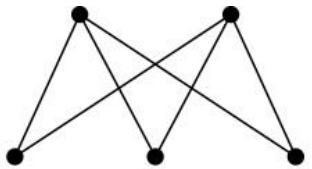
Example: Show that C_3 is not bipartite.

Solution: If we divide the vertex set of C_3 into two nonempty sets, one of the two must contain two vertices. But in C_3 every vertex is connected to every other vertex. Therefore, the two vertices in the same partition are connected. Hence, C_3 is not bipartite.

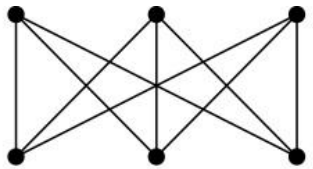
Complete Bipartite Graphs

Definition: A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

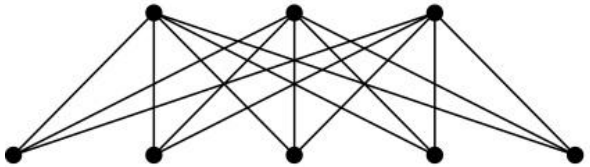
Example: We display four complete bipartite graphs here.



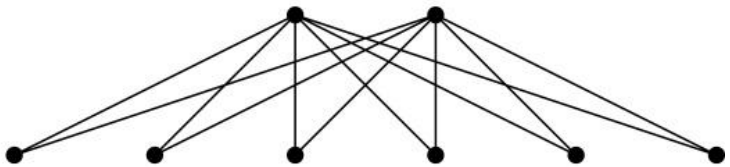
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$

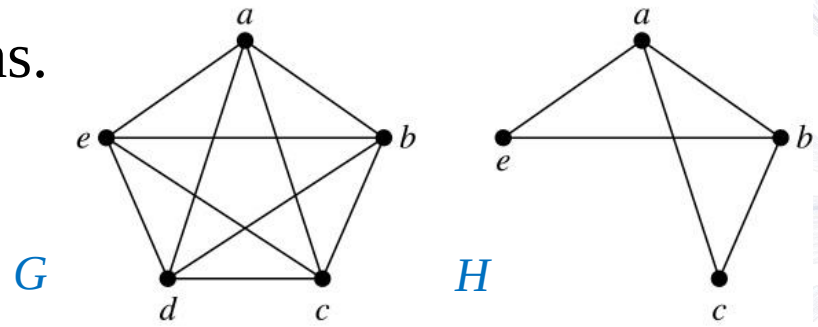


$K_{2,6}$

New Graphs from Old

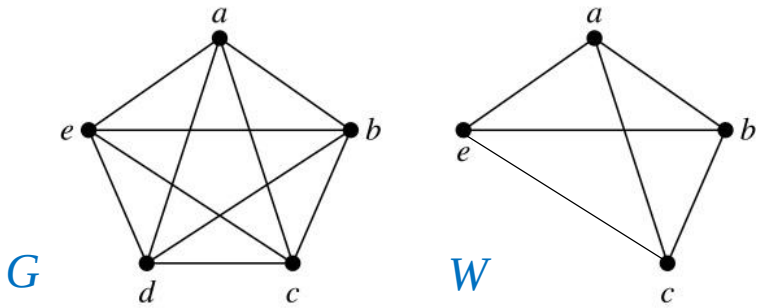
Definition: A *subgraph of a graph* $G = (V,E)$ is a graph (W,F), where $W \subset V$ and $F \subset E$. A subgraph H of G is a proper subgraph of G if $H \neq G$.

Example: Here we show K_5 and one of its subgraphs.



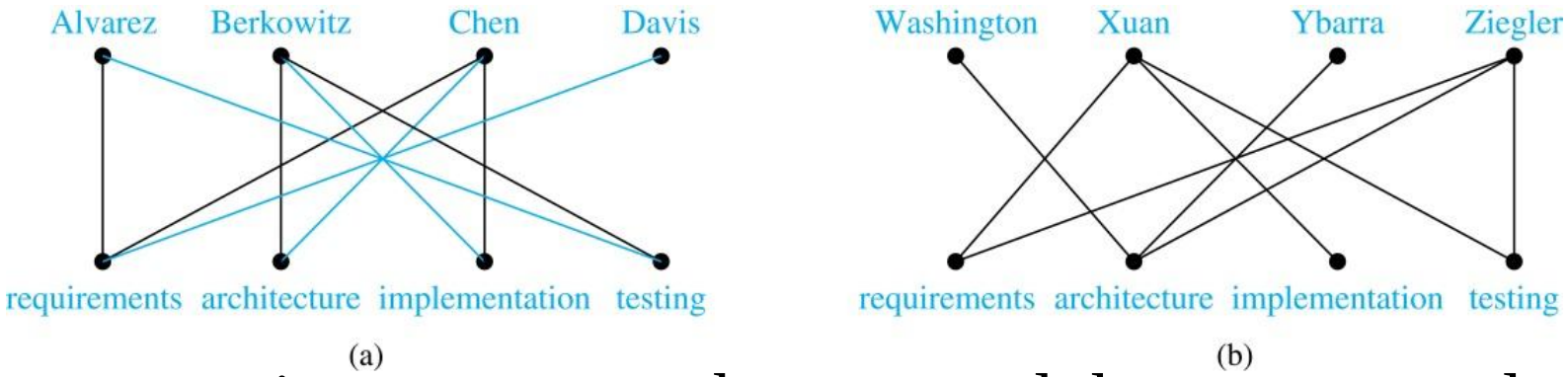
Definition: Let $G = (V, E)$ be a simple graph. The *subgraph induced* by a subset W of the vertex set V is the graph (W,F), where the edge set F contains an edge in E if and only if both endpoints are in W.

Example: Here we show K_5 and the subgraph induced by $W = \{a,b,c,e\}$.



Bipartite Graphs and Matchings

- Bipartite graphs are used to model applications that involve matching the elements of one set to elements in another, for example:
- *Job assignments* - vertices represent the jobs and the employees, edges link employees with those jobs they have been trained to do. A common goal is to match jobs to employees so that the most jobs are done

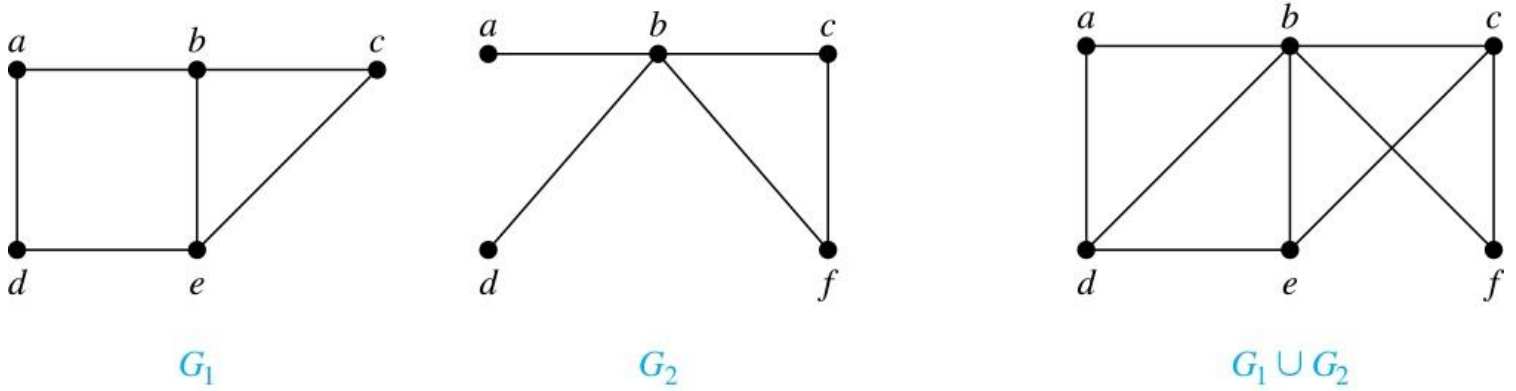


- *Marriage* - vertices represent the men and the women and edges link a a man and a woman if they are an acceptable spouse. We may wish to find the largest number of possible marriages

New Graphs from Old (continued)

Definition: The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Example:



厦門大學信息学院
School of Informatics Xiamen University
(特色化示范性软件学院)
(National Characteristic Demonstration Software School)

Q&A